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Transmission, reflection and localization in a random medium with absorption or gain

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Abstract
We study reflection and transmission of waves in a random tight-binding system with absorption or gain for weak disorder, using a scattering matrix formalism. Our aim is to discuss analytically the effects of absorption or gain on the statistics of wave transport. Treating the effects of absorption or gain exactly in the limit of no disorder allows us to identify short- and long-length regimes relative to absorption or gain lengths, where the effects of absorption/gain on statistical properties are essentially different. In the long-length regime, we find that a weak absorption or a weak gain induce identical statistical corrections in the inverse localization length, but lead to different corrections in the mean reflection coefficient. In contrast, a strong absorption or a strong gain strongly suppress the effect of disorder in identical ways (to leading order), both in the localization length and in the mean reflection coefficient.

1. Introduction

In this paper we study analytically the coherent reflection and transmission of waves in an active one-dimensional disordered system which either absorbs or amplifies the waves. Our model is the familiar single-band tight-binding model with random site energies (Anderson model) including additional fixed positive or negative imaginary parts describing absorption or amplification. As is well-known, the introduction of the imaginary potential destroys the time-reversal symmetry of the system.

The electronic model with absorption may describe annihilation of electrons via electron–hole recombinations acting as a complex optical potential in a nearly compensated semiconductor. The amplification model is, of course, meaningless for electrons whose fermionic character forbids the presence of more than one electron at a given spatial location.

On the other hand, the tight-binding model with absorption or amplification due to stimulated emission may be appropriate for describing the localization of light waves in active photonic band-gap crystals, characterized by a periodic variation of the dielectric constant. In particular, the interplay of the (phase coherent) amplification of light waves with the process of coherent scattering by random inhomogeneities leading to localization [1, 2] is of current interest for random lasers [3].

A considerable amount of theoretical work related to the statistics of the transmittance and of the reflectance of random systems with absorption or gain has already been published [4–15]. We feel, however, that important aspects of the effects of absorption or gain on the transmission properties of the random system studied below have not received sufficient attention in previous work.

We refer, in particular, to the form of the localization length ξ for large lengths L of a random chain. For weak disorder, this is found to be given by [6, 9]

$$\frac{1}{\xi} = \frac{1}{l_u} + \frac{1}{\xi_0}, \qquad u = \mathrm{a} \text{ or } \mathrm{g}, \tag{1}$$

where ξ_0 is the localization length of the system in the absence of absorption or gain (amplification), and l_a and l_g are the absorption and gain lengths of a perfect system, respectively. In the tight-binding model, (1) is expected to be obtained by studying the transmittance of the disordered sample described by the Schrödinger equation

$$[E - (\varepsilon_n + \mathrm{i}\eta)]\varphi_n = \varphi_{n+1} + \varphi_{n-1}, \qquad n = 1, 2, \ldots, N, \tag{2}$$

where φ_n are the wavefunction amplitudes at sites $n = 1, 2, \ldots, N$, of spacing a, of the disordered sample of length $L = Na$. The ε_n are the random site energies, to which one adds a fixed non-Hermitian term $\mathrm{i}\eta$ describing absorption for $\eta > 0$ and amplification for $\eta < 0$. The energies E, ε_n and η are in units of a fixed nearest-neighbour hopping energy V. The random chain is connected at both ends to semi-infinite perfect leads ($\varepsilon_n = 0$) with $\eta = 0$, whose sites are positioned at $n = 0, -1, -2, \ldots$ and $n = N+1, N+2, \ldots$, respectively. For the perfect tight-binding chain with absorption or gain, Datta [12] has derived

$$l_\mathrm{g} = l_\mathrm{a} = \frac{1}{|\eta|}, \tag{3}$$

while numerical studies of the random tight-binding model by Gupta, Joshi and Jayannavar [11] and by Jiang and Soukoulis [13] support equation (1) for small η of either sign, with $1/\xi_0$ given by the familiar Thouless expression for weak site-energy disorder. A feature of (1) that is generally regarded as paradoxical is the fact that for amplification it leads to the suppression of transmittance for large L, as in the case of absorption. What is more, according to equation (1), when disorder is present the suppression for amplification occurs at exactly the same rate as for absorption [9]. This surprising feature has been an important incentive for developing a more comprehensive analytic treatment of the statistics of wave transport in the presence of absorption or amplification. Indeed, an important drawback of equation (1) is that it completely ignores effects of absorption or gain on the statistics of the transmission coefficient from which $\frac{1}{\xi}$ is obtained. Our aim is to remedy this defect in the framework of a detailed analysis for weak disorder of the tight-binding system defined by equation (2).

An important feature of our approach below is an exact treatment of the effect of absorption or gain, as done previously by Datta [12] for a non-disordered (pure) system. This allows us to clearly identify and discuss short- and long-length regimes relative to the absorption (amplification) length in (1).

The study of transmission and reflection in random one-dimensional media with absorption or gain was initiated, and later pursued actively, using invariant imbedding equations [4–10]. These equations are coupled non-linear differential equations for the reflection and transmission amplitudes of plane waves incident to the right of a continuous medium occupying the domain $0 \leqslant x \leqslant L$ of the x-direction[1]. As was shown recently [16], the invariant imbedding equations

[1] The random potential is assumed to be inhomogeneous in the x-direction only and includes a fixed imaginary part describing absorption ($\eta > 0$) or gain ($\eta < 0$).

follow for weak disorder from the long-wavelength continuum limit of (2) for a disordered chain embedded in an infinite perfect chain. We recall that the invariant imbedding equations were originally derived as an exact consequence of the Helmholtz equation for the propagation of the electric field in a dielectric medium [4, 17][2]. For later discussion, we also recall the important early result for the so-called short-length localization length [4, 5],

$$\frac{1}{\xi} = \eta + \frac{1}{\xi_0}, \tag{4}$$

which indicates that, at short lengths, transmittance increases with L for amplification, if $l_a < \xi_0$.

The paper is organized as follows. In section 2 we derive exact expressions for the scattering matrix elements (transmission and reflection amplitudes) in terms of transfer matrices for the tight-binding equation (1) for weak disorder. In section 3 we discuss our explicit analytic results for the averaged logarithmic transmission coefficients (localization lengths) in the short- and long-length regimes. We also discuss the mean logarithmic reflection coefficient for large lengths. We recall that the distribution of the reflection coefficient is important in the context of random lasers [7, 8]. Some concluding remarks follow in section 4.

2. Transfer matrix analysis

We start our analysis by rewriting (2) in terms of a transfer matrix for a site n:

$$\begin{pmatrix} \varphi_{n+1} \\ \varphi_n \end{pmatrix} = \hat{P}_n \begin{pmatrix} \varphi_n \\ \varphi_{n-1} \end{pmatrix}, \qquad \hat{P}_n = \begin{pmatrix} E - \varepsilon_n - \mathrm{i}\eta & -1 \\ 1 & 0 \end{pmatrix}. \tag{5}$$

The analogous equation for sites in the perfect leads involves the transfer matrix $\hat{P}^0$ obtained by letting $\varepsilon_n = \eta = 0$ in (5). We wish to study the scattering (reflection and transmission) of (Bloch) plane wave states of the leads by the disordered segment of length $L \equiv N$ (with $a = 1$). For this purpose, it is necessary to perform a similarity transformation of (5) to the basis of the Bloch wave solutions $\varphi_n \sim \mathrm{e}^{\pm \mathrm{i}kn}$ for the leads. The eigenvectors of $\hat{P}^0$ are of the form $\vec{u}_\pm = \binom{\mathrm{e}^{\pm \mathrm{i}k}}{1}$ with eigenvalues $\mathrm{e}^{\pm \mathrm{i}k}$ obeying the equation

$$E = 2\cos k, \tag{6}$$

which defines the tight-binding energy band. As usual, we choose k positive, $0 \leqslant k \leqslant \pi$, so that, for example, $\mathrm{e}^{\mathrm{i}kn}$ corresponds to a wave propagating from left to right on the lattice of (2). The similarity transformation of $\hat{P}_n$ to the Bloch wave basis is defined by the matrix $\hat{U} = (\vec{u}_+, \vec{u}_-)$ and leads to

$$\hat{Q}_n = \hat{U}^{-1}\hat{P}_n\hat{U} = \hat{Q}_n^0 + \hat{Q}_n^1, \tag{7}$$

where

$$\hat{Q}_n^0 = \begin{pmatrix} (1-b)\mathrm{e}^{\mathrm{i}k} & -b\mathrm{e}^{-\mathrm{i}k} \\ b\mathrm{e}^{\mathrm{i}k} & (1+b)\mathrm{e}^{-\mathrm{i}k} \end{pmatrix}, \qquad \hat{Q}_n^1 = \mathrm{i}b_n \begin{pmatrix} \mathrm{e}^{\mathrm{i}k} & \mathrm{e}^{-\mathrm{i}k} \\ -\mathrm{e}^{\mathrm{i}k} & -\mathrm{e}^{-\mathrm{i}k} \end{pmatrix}, \tag{8}$$

[2] In the appendix of [10], a new simple derivation of the invariant imbedding equations from the Helmholtz equation was presented. This derivation depends on the identification of $\mathrm{d}y(L, L)/\mathrm{d}L$ (where $y(x, L) = \frac{\partial}{\partial x} \ln E(x, L)$, with $E(x, L)$ the electric wavefield) with $\left(\frac{\partial y(x,L)}{\partial x}\right)_{x=L}$ at the edge $x = L$ of the layer extending between $x = 0$ and L. In [10], this identification was regarded as 'a basic assumption of the invariant imbedding procedure'. Actually, it turns out that this is not an assumption at all, since it may be shown, using the exact identity $\frac{\partial E(x,L)}{\partial L} = a(L)E(x, L)$, $a(L) = \mathrm{i}k_0\left[1 + \frac{1}{2}\eta(L)E(L, L)\right]$, derived in [4], that $\left(\frac{\partial y(x,L)}{\partial L}\right)_{x=L} = 0$. In conclusion, the derivation of the invariant imbedding equations in [10] is exact.

and

$$b = \frac{\eta}{2\sin k}, \qquad b_n = \frac{\varepsilon_n}{2\sin k}. \tag{9}$$

The transfer matrix of the disordered segment of length N is the product of transfer matrices associated with the individual sites:

$$\hat{Q} = \prod_{n=1}^{N} \hat{Q}_n. \tag{10}$$

We now recall the precise relationship between the transfer matrix elements $(\hat{Q})_{ij} \equiv Q_{ij}$ and the reflection and transmission amplitudes r^{+-} and t^{--} and r^{-+} and t^{++} for waves incident at the right and at the left of the disordered system, respectively. The reflection and transmission amplitudes define the scattering matrix $\hat{S}$, which expresses outgoing wave amplitudes at the left (O) and at the right (O') of the disordered segment in terms of ingoing ones, (I) and (I') [18]:

$$\begin{pmatrix} O \\ O' \end{pmatrix} = \begin{pmatrix} r^{-+} & t^{--} \\ t^{++} & r^{+-} \end{pmatrix} \begin{pmatrix} I \\ I' \end{pmatrix}. \tag{11}$$

The transfer matrix $\hat{Q}$, on the other hand, gives the Bloch wave amplitudes at the right end of the disordered section in terms of the amplitudes at the left end:

$$\begin{pmatrix} O' \\ I' \end{pmatrix} = \hat{Q} \begin{pmatrix} I \\ O \end{pmatrix}, \tag{12}$$

whose transformation to a form analogous to (11) yields:

$$\begin{pmatrix} O \\ O' \end{pmatrix} = \frac{1}{Q_{22}} \begin{pmatrix} -Q_{21} & 1 \\ \det \hat{Q} & Q_{12} \end{pmatrix} \begin{pmatrix} I \\ I' \end{pmatrix}, \tag{13}$$

which leads to the desired expressions of transmission and reflection amplitudes in terms of the transfer matrix elements $Q_{ij} = (\hat{Q})_{ij}$:

$$t^{--} = \frac{1}{Q_{22}}, \qquad t^{++} = \left(\det \hat{Q}\right) t^{--}, \tag{14}$$

$$r^{+-} = \frac{Q_{12}}{Q_{22}}, \qquad r^{-+} = -\frac{Q_{21}}{Q_{22}}. \tag{15}$$

From (10), it follows that the determinant of $\hat{Q}$ is the product of the determinants of the exact transfer matrices $\hat{Q}_n$ associated with the individual sites n of the disordered segment of N sites, of lengths Na. Now, from (7) to (8), we find that $\det \hat{Q}_n = 1, n = 1, 2, \ldots, N$, so that

$$\det \hat{Q} = 1. \tag{16}$$

From (14), it then follows that

$$t^{++} = t^{--} \equiv t = \frac{1}{Q_{22}}, \tag{17}$$

for any realization of the disorder and for any strength of the imaginary potential.

As usual, we assume that the random site energies are identically distributed independent Gaussian variables with mean zero and correlation

$$\langle \varepsilon_n \varepsilon_m \rangle = \varepsilon_0^2 \delta_{m,n}. \tag{18}$$

For weak disorder, we shall expand the matrix functional $\hat{Q}$ to linear order in the random site energies. We note that, since the energies of neighboring sites are uncorrelated (equation (18)), the second-order terms in the expansion of (10) may be omitted, since they will not contribute to averages over the disorder at the order ε_0^2. Thus, restricting to first order in the expansion of (10), we obtain [18]

$$\hat{Q} = \left(\hat{Q}_n^0\right)^N + \sum_{m=1}^{N} \left(\hat{Q}_n^0\right)^{m-1} \hat{Q}_m^1 \left(\hat{Q}_n^0\right)^{N-m} \equiv \hat{Q}^0 + \hat{Q}^1. \tag{19}$$

The transfer matrix product $(\hat{Q}_n^0)^N$ for the medium in the absence of disorder may readily be evaluated in closed form. This will allow us to discuss analytically the reflection and transmission properties of a perfectly amplifying or absorbing system, as done earlier by Datta [12] using a slightly different procedure. We write

$$\left(\hat{Q}_n^0\right)^N = \hat{V}\left(\hat{V}^{-1}\hat{Q}_n^0\hat{V}\right)^N \hat{V}^{-1}, \qquad \hat{V} = (\vec{v}_+, \vec{v}_-), \tag{20}$$

where $\hat{V}$ is the diagonalizing matrix formed by the eigenvectors

$$\vec{v}_+ = \begin{pmatrix} \frac{b\mathrm{e}^{-\mathrm{i}k}}{(1-b)\mathrm{e}^{\mathrm{i}k}-\mathrm{e}^{\mathrm{i}q}} \\ 1 \end{pmatrix}, \qquad \vec{v}_- = \begin{pmatrix} \frac{b\mathrm{e}^{-\mathrm{i}k}}{(1-b)\mathrm{e}^{\mathrm{i}k}-\mathrm{e}^{-\mathrm{i}q}} \\ 1 \end{pmatrix}, \tag{21}$$

(with $\det \hat{V} = \frac{2\mathrm{i}\mathrm{e}^{-\mathrm{i}k}}{b^2}\sin q$) of $\hat{Q}_n^0$ corresponding to eigenvalues $\mathrm{e}^{\mathrm{i}q}$ and $\mathrm{e}^{-\mathrm{i}q}$, respectively. These eigenvalues are defined in terms of complex functions $\cos q$ and $\sin q$ by

$$\begin{aligned} &\mathrm{e}^{\pm\mathrm{i}q} = \cos q \pm \mathrm{i}\sin q, \\ &\cos q = \cos k - \mathrm{i}b\sin k, \qquad \sin q = \sqrt{1-\cos^2 q}. \end{aligned} \tag{22}$$

The explicit expression of $\left(\hat{Q}_n^0\right)^m$ obtained from (20) to (22) is

$$\left(\hat{Q}_n^0\right)^m = \begin{pmatrix} A_m & B_m \\ C_m & D_m \end{pmatrix}, \tag{23}$$

where (with $\mathrm{e}^{\pm\mathrm{i}k} \equiv d_\pm$)

$$A_m = \frac{1}{\sin q}[(1-b)d_+\sin qm - \sin q(m-1)], \tag{24}$$

$$D_m = -\frac{1}{\sin q}[(1-b)d_+\sin qm - \sin q(m+1)], \tag{25}$$

$$B_m = -bd_-\frac{\sin qm}{\sin q}, \tag{26}$$

$$C_m = bd_+\frac{\sin qm}{\sin q}. \tag{27}$$

Note that, as expected, the transfer matrix of the non-disordered system with absorption or gain defined by (24)–(27) obeys

$$A_N D_N - B_N C_N = \cos^2 qN + \sin^2 qN = 1, \tag{28}$$

and from (15) and (26) and (27) it follows that, in this case, the reflection coefficients are equal

$$|r^2| \equiv |r^{+-}|^2 = |r^{-+}|^2 = \frac{|B_N|^2}{|D_N|^2}, \qquad \varepsilon_m = 0, \quad m = 1, 2, \ldots, N. \tag{29}$$

Finally, by inserting (8) and (23) in (19), we obtain the explicit form of the transfer matrix of the disordered section to first order:

$$\hat{Q} = \begin{pmatrix} A_N & B_N \\ C_N & D_N \end{pmatrix} + \begin{pmatrix} Q_{11}^1 & Q_{12}^1 \\ Q_{21}^1 & Q_{22}^1 \end{pmatrix}, \tag{30}$$

where

$$Q_{11}^{1} = \mathrm{i}\sum_{m=1}^{N} b_m(A_{m-1} - B_{m-1})(d_+A_{N-m} + d_-C_{N-m}), \tag{31}$$

$$Q_{22}^{1} = \mathrm{i}\sum_{m=1}^{N} b_m(C_{m-1} - D_{m-1})(d_+B_{N-m} + d_-D_{N-m}), \tag{32}$$

$$Q_{12}^{1} = \mathrm{i}\sum_{m=1}^{N} b_m(A_{m-1} - B_{m-1})(d_+B_{N-m} + d_-D_{N-m}), \tag{33}$$

$$Q_{21}^{1} = \mathrm{i}\sum_{m=1}^{N} b_m(C_{m-1} - D_{m-1})(d_+A_{N-m} + d_-C_{N-m}). \tag{34}$$

The lowest-order effect of the disorder in the mean transmission and reflection coefficients is obtained by expanding (15) and (17) to second order in the disorder, using (30) and noting that the first-order averages vanish. However, in the case of the reflection coefficients, which are asymptotically independent of length in the absence of disorder [12, 13], it is apt to focus on the simpler averages $\langle \ln |r^{\pm\mp}|^2\rangle$. On the other hand, the study of the mean logarithm of the transmission coefficient is of special interest, since it is related asymptotically to the localization length

$$\frac{1}{\xi_\pm} = -\lim_{N\to\infty} \frac{\langle \ln |t^{\pm\pm}|^2\rangle}{2N}, \tag{35}$$

which is a self-averaging quantity in the absence of absorption or gain [19]. From (15), (17) and (30), we obtain successively for the quantities of interest, to second order in the disorder,

$$\langle \ln |t|^2\rangle = (-\ln D_N + \text{c.c.}) + \frac{1}{2}\left(\frac{\langle (Q_{22}^1)^2\rangle}{D_N^2} + \text{c.c.}\right), \tag{36}$$

$$\langle \ln |r^{+-}|^2\rangle = \left(\ln\frac{B_N}{D_N} + \text{c.c.}\right) - \frac{1}{2}\left[\left(\left\langle\left(\frac{Q_{12}^1}{B_N}\right)^2\right\rangle - \left\langle\left(\frac{Q_{22}^1}{D_N}\right)^2\right\rangle\right) + \text{c.c.}\right], \tag{37}$$

$$\langle \ln |r^{-+}|^2\rangle = \left(\ln\frac{C_N}{D_N} + \text{c.c.}\right) - \frac{1}{2}\left[\left(\left\langle\left(\frac{Q_{21}^1}{C_N}\right)^2\right\rangle - \left\langle\left(\frac{Q_{22}^1}{D_N}\right)^2\right\rangle\right) + \text{c.c.}\right]. \tag{38}$$

We close this section by demonstrating the equivalence of our results for the transmission and reflection coefficients for the perfect absorbing or amplifying system ($\varepsilon_m = 0$) and the corresponding results obtained earlier by Datta [12] at the band centre. For this purpose, we identify the parameters $\mathrm{e}^{\mathrm{i}q}$ and $\mathrm{e}^{-\mathrm{i}q}$ defined above respectively with the quantities $\mathrm{i}\mathrm{e}^{-\xi}$ and $-\mathrm{i}\mathrm{e}^{\xi}$ involving the parameter ξ introduced by Datta via the substitution $\sinh\xi = \frac{\eta}{2}$. By transforming equation (5) of [12] for $|t|^2$ for even N in terms of the variable q, we get

$$|t|^2 = \frac{\sin^2 q}{(-\mathrm{i}\sin Nq + \sin q\cos Nq)^2}, \tag{39}$$

which coincides with the expression obtained by substituting (25) for $k = \pi/2$ and $m = N$ in the definition (17) of $|t^{--}|$. Similarly, the transformation of equation (6) of [12] for odd N again yields (39), obtained from (17) and (25) above. The equations (7) and (8) of Datta [12] for the reflection coefficient $|r|^2$ for even and odd N, respectively, reduce similarly to the corresponding expressions obtained from (25) to (26) and (29). Clearly, the advantage of the present treatment is that it condenses distinct expressions for even and odd N in Datta's analysis into a single expression for any one of the amplitudes coefficients in (15), (17). This is clearly useful, particularly for handling the more cumbersome general expressions for the effect of weak disorder.

3. Detailed results for $E = 0$

For simplicity, and as in most previous work for the tight-binding model [11–13], we restrict the analytical calculations and results in this section to the band centre, $E = 0 (k = \pi/2)$. At the band centre, the pure system transfer matrix elements (24)–(27) take the simple forms

$$A_m = u_+ \mathrm{e}^{\mathrm{i}qm} - u_- \mathrm{e}^{-\mathrm{i}qm}, \tag{40}$$

$$D_m = -u_- \mathrm{e}^{\mathrm{i}qm} + u_+ \mathrm{e}^{-\mathrm{i}qm}, \tag{41}$$

$$B_m = C_m = v(\mathrm{e}^{\mathrm{i}qm} - \mathrm{e}^{-\mathrm{i}qm}), \qquad m = 1, 2, \ldots, N, \tag{42}$$

where

$$u_\pm = \frac{1}{2}\left(\frac{1}{\sqrt{1+b^2}} \pm 1\right), \qquad v = \frac{b}{2\sqrt{1+b^2}}, \tag{43}$$

$$\mathrm{e}^{\pm\mathrm{i}q} = (\pm\mathrm{i})(\sqrt{1+b^2} \mp b). \tag{44}$$

For the pure tight-binding system with absorption or gain ($\varepsilon_m = 0, m = 1, 2 \ldots N$), the transmittances and reflectances, for both directions of incidence, are given exactly for any band energy E and for any length $L = N$ by substituting the closed expressions (24)–(27) for the transfer matrix elements into the definitions (15) and (17). Exact results for $|t|^2$ and $|r|^2$ for the perfect system with absorption or gain, for $E = 0$, have been discussed by Datta [12], and more extensive numerical results that include the additional effect of a weak disorder on the averaged logarithmic transmittance have been presented by Jiang and Soukoulis [13]. Special attention has been paid in [12, 13] to the domain of intermediate lengths (in particular, the critical length L_c) where the transmittance of an amplifying system changes from an initial growth at short lengths to an exponential decay at long lengths.

In the following, we discuss detailed results for transmittance and reflectance in the framework of the general analytic treatment for weak disorder in section 2. We shall consider successively the short- and the long-length domains defined below. Our consistent treatment of the effect of weak disorder in the framework of an exact analysis of absorption or gain at zeroth order leads to the identification of the important effects induced by absorption or amplification in the statistics of wave transport.

3.1. Short lengths

For a fixed magnitude of the absorption/amplification parameter b, the short-length domain is defined by

$$N|b| \ll 1, \tag{45}$$

or, equivalently, $L \ll l_0$, where $l_0 = 1/b$ for $b > 0$ is the absorption length (in units of a) and $l_0 = -1/b$ for $b < 0$ is the amplification length. We wish to obtain the logarithmic transmittance in the limit (45), which determines the short-length localization length. For this purpose, we use the following approximations of (40)–(42) valid to lowest order, for small $|b|$ and small $m|b|$:

$$A_m = \mathrm{i}^m(1 - mb), \tag{46}$$

$$D_m = (-\mathrm{i})^m(1 + mb), \tag{47}$$

$$B_m = C_m = \frac{b}{2}(\mathrm{i}^m + (-\mathrm{i})^m). \tag{48}$$

Note that these expressions would not be sufficient for discussing the reflection coefficients whose explicit forms differ for even and odd N and require the inclusion of higher orders in

$ma \equiv m$ for B_m and C_m. For the sake of brievety, we omit discussing the short-length reflection coefficients in more detail. We substitute (46)–(48) in the expression (32) for $E = 0$, which we then insert in (36). After averaging over the disorder, using (18), and performing the remaining geometric sums over sites, we obtain the following final results:

$$\langle \ln |t|^2 \rangle = -2bN - \frac{\varepsilon_0^2}{4}(1-4b)N + \mathrm{O}(b^2N^2). \tag{49}$$

The short-length localization length obtained from (35) and (49), namely

$$\frac{1}{\xi} = b + \frac{\varepsilon_0^2}{8}(1-4b), \tag{50}$$

yields

$$\frac{1}{\xi} = \frac{1}{l_0} + \frac{1}{\xi_0} - \frac{4}{\xi_0 l_0}, \tag{51}$$

for absorption, and

$$\frac{1}{\xi} = -\frac{1}{l_0} + \frac{1}{\xi_0} + \frac{4}{\xi_0 l_0}, \tag{52}$$

for amplification, where

$$\xi_0 = \frac{8}{\varepsilon_0^2}. \tag{53}$$

Note that (53) is the exact perturbation expression (for $E = 0$) of the localization length for weak disorder, for $b = 0$. Indeed it coincides with the well-known exact result, $\xi_0 = 96W^{-2}\sin^2 k$, obtained by Thouless [20], if the variance $W^2/12$ of the rectangular distribution of width W of site energies in [20] is replaced by the gaussian mean square ε_0^2. The first two terms in (51)–(52) agree with the form of the short-length localization lengths derived previously from invariant imbedding [4, 5, 10].

Finally, it is useful to clarify the general meaning of short- and long-length localization lengths in the framework of our weak disorder analysis. We recall that the perturbation treatment of disorder in section 2 is valid for

$$N\varepsilon_0^2 \ll 1. \tag{54}$$

This implies, in particular, that the localization length (53) is valid for values $\varepsilon_0^2 \to 0$ such that the limit (54) embraces asymptotically large N, for which the localization length is defined in (35). Similarly, the short-length localization length (50) in the presence of absorption or gain is a true localization length only if it corresponds to the limit of asymptotically large N in (35). Thus, if $|b| < \varepsilon_0^2$, this limit is obtained for $|b| \to 0$ (since $N|b| < N\varepsilon_0^2 \ll 1$) while, if $|b| > \varepsilon_0^2$, it is obtained by letting $\varepsilon_0^2 \to 0$. Now, for $|b| < \varepsilon_0^2$, (54) automatically implies (45), in which case the short-length expressions (51) and (52) give the true localization lengths. On the other hand, for $|b| > \varepsilon_0^2$, two possibilities exist for the localization lengths:

- if, for asymptotic lengths obeying (54) ($\varepsilon_0^2 \to 0$), one also has $N|b| \ll 1$, then the localization lengths are clearly given by the 'short-length' expressions (51) and (52). This happens for values $|b| \geqslant \varepsilon_0^2$ sufficiently close to ε_0^2.
- if, for asymptotic lengths (54), the long-length condition

$$N|b| \gg 1, \tag{55}$$

is fulfilled, then the localization lengths are given by the equations (64*a*), (64*b*) and their limiting forms (66*a*), (66*b*) and (68*a*), (68*b*) in section 3.2 below. This situation exists for $|b|$-values sufficiently larger than ε_0^2.

3.2. Long lengths

The transfer matrix elements (40)–(42) of a perfect system depend on the imaginary exponentials $e^{\pm iqN}$, which for $|b| \ll 1$ are given by

$$e^{\pm iqN} = (\pm i)^N e^{\mp N(b+\frac{b^3}{3}+\cdots)}, \tag{56}$$

where e^{iqN} grows exponentially for $b < 0$ (amplification) and e^{-iqN} grows for $b > 0$ (absorption) in the long-length regime, $N \gg |b|^{-1}$ (55). We first discuss the detailed form of the logarithmic transmission coefficient, $\ln|t|^2 = \ln t + \text{c.c.}$, and of the reflection coefficient $|r^{\pm\mp}|^2$ given by (17), (15) and (30)–(34) in terms of the transfer matrix elements A_N, B_N, C_N, D_N in (40)–(42). Retaining only the leading exponential terms at long lengths for absorption and amplification, respectively, we obtain successively

$$\ln|t|^2 = -2\left|\left(b + \frac{b^3}{3}\right)\right| N + \mathrm{O}(N^{-1}), \tag{57}$$

for both signs of b, and

$$|r^{+-}|^2 = |r^{-+}|^2 \simeq \frac{b^2}{4}, \qquad b > 0, \tag{58}$$

$$|r^{+-}|^2 = |r^{-+}|^2 \simeq \frac{4}{b^2}, \qquad b < 0. \tag{59}$$

The main feature of these results is that $\ln|t|^2$ is decreasing at large L for amplification as well as for absorption, in agreement with previous studies [6, 9, 11–13].

Next, we consider the effect of weak disorder at $E = 0$ in the mean logarithmic transport coefficients (36)–(38) involving zeroth- and first-order transfer matrix elements defined in (40)–(42) and (31)–(34). Using (18), the averages of the various quadratic forms in first-order transfer matrix elements in (36)–(38) reduce to simple sums over lattice sites m of the products of two terms of the form $M_{m-1} - N_{m-1}$ corresponding to the site $m-1$ multiplied by the product of two terms of the form $P_{N-m} - R_{N-m}$ corresponding to the site $N - m$ (with M, N, P, R representing elements, distinct or not, of the set of transfer matrix elements A, B, C, D of the pure system). Using (40)–(42), we approximate the mth term in a given sum by the contribution that is independent of m, which yields the leading effect proportional to N for any of the sums involved (the contributions ignored in this approximation are readily shown to be of relative order $\frac{1}{N}$). For the averages of products of first-order transfer matrix elements entering into (36)–(38), we thus obtain the following results valid at $E = 0$, for any sign of b:

$$\left\langle \left(Q_{22}^1\right)^2 \right\rangle = \frac{N\varepsilon_0^2}{4}\left[(u_- + v)^2 e^{iq(N-1)} + [(u_+ + v)^2 e^{-iq(N-1)}\right]^2, \tag{60}$$

$$\left\langle \left(Q_{12}^1\right)^2 \right\rangle = \frac{N\varepsilon_0^2}{4}\left[(u_+ - v)(u_- + v) e^{iq(N-1)} + [(u_+ + v)(u_- - v) e^{-iq(N-1)}\right]^2, \tag{61}$$

$$\left\langle \left(Q_{21}^1\right)^2 \right\rangle = \langle (Q_{12}^1)^2 \rangle. \tag{62}$$

From (62) and (37) and (38), it follows that

$$\langle \ln|r^{+-}|^2 \rangle = \langle \ln|r^{-+}|^2 \rangle \equiv \langle \ln|r|^2 \rangle, \tag{63}$$

for both signs of b.

Next, we insert (60)–(62), together with (41) and (42), in (35)–(38) and simplify the resulting expressions by retaining in each one of them only the leading exponential terms for $N|b| \gg 1$, successively for $b > 0$ and $b < 0$. In this way, we obtain the following exact expressions valid for arbitrary $|b|$ larger than ε_0^2 and such that $N|b| \gg 1$:

$$\frac{1}{\xi} = \frac{1}{2N}\ln|e^{-iqN}|^2 - \frac{\varepsilon_0^2}{8}\frac{(u_+ + v)^4}{u_+^2} e^{2iq}, \tag{64a}$$

$$\langle \ln |r|^2 \rangle = \ln \left(\frac{v}{u_+} \right)^2 - \frac{\varepsilon_0^2 N}{4} (u_+ + v)^2 \left[\frac{(u_- - v)^2}{v^2} + \frac{(u_+ + v)^2}{u_+^2} \right] \mathrm{e}^{2\mathrm{i}q}, \qquad b > 0; \quad (65a)$$

$$\frac{1}{\xi} = \frac{1}{2N} \ln |\mathrm{e}^{\mathrm{i}qN}|^2 - \frac{\varepsilon_0^2}{8} \frac{(u_- + v)^4}{u_-^2} \mathrm{e}^{-2\mathrm{i}q}, \quad (64b)$$

$$\langle \ln |r|^2 \rangle = \ln \left(\frac{v}{u_-} \right)^2 - \frac{\varepsilon_0^2 N}{4} (u_- + v)^2 \left[\frac{(u_+ - v)^2}{v^2} + \frac{(u_- + v)^2}{u_-^2} \right] \mathrm{e}^{-2\mathrm{i}q}, \qquad b < 0. \quad (65b)$$

For weak absorption/amplification $|b| \ll 1$ (with, however, $N|b| \gg 1$) we expand (64a), (65a) and (64b), (65b) in powers of b, using (43), (44) and (56). To order b^2 in the effects of the disorder, we finally obtain

$$\frac{1}{\xi} = b + \frac{b^3}{3} + \frac{\varepsilon_0^2}{8}(1 - b^2), \quad (66a)$$

$$\langle \ln |r|^2 \rangle = \ln \left(\frac{b^2}{4} \right) - \frac{b^2}{2} + \frac{\varepsilon_0^2 b^2}{2} N, \qquad b > 0; \quad (67a)$$

$$\frac{1}{\xi} = -\left(b + \frac{b^3}{3} \right) + \frac{\varepsilon_0^2}{8}(1 - b^2), \quad (66b)$$

$$\langle \ln |r|^2 \rangle = \ln \left(\frac{4}{b^2} \right) + \frac{b^2}{2}, \qquad b < 0. \quad (67b)$$

On the other hand, for strong absorption/amplification, $|b| \gg 1$, we obtain successively from (64a), (65a) and (64b), (65b), to the orders indicated, using (44),

$$\frac{1}{\xi} = -\ln(2b) - \frac{1}{4b^2} + \frac{\varepsilon_0^2}{8b^2}, \quad (68a)$$

$$\langle \ln |r|^2 \rangle = -\frac{2}{b} \left(1 + \frac{1}{6b^2} \right) + \frac{\varepsilon_0^2 N}{2b^2} \left(1 - \frac{1}{|b|} \right), \qquad b > 0; \quad (69a)$$

$$\frac{1}{\xi} = \ln(2|b|) + \frac{1}{4b^2} + \frac{\varepsilon_0^2}{8b^2}; \quad (68b)$$

$$\langle \ln |r|^2 \rangle = \frac{2}{b} \left(1 - \frac{1}{3b^2} \right) + \frac{\varepsilon_0^2 N}{2b^2} \left(1 + \frac{1}{|b|} \right), \qquad b < 0, \quad (69b)$$

using the expansions of (43) and (44) in powers if $\frac{1}{|b|}$.

Our detailed results (66a), (66b) and (68a), (68b) for localization lengths and (67a), (67b) and (69a), (69b) for logarithmic reflection coefficients display remarkable new features related to the effects induced by absorption/amplification in statistical averages over the disorder. We recall that our results are valid at asymptotic lengths for values $|b| > \varepsilon_0^2$, but not too close to ε_0^2 (see the discussions in section 3.1 above). In the absence of induced statistical effects, the results for inverse localization lengths coincide with the previously known results (1), (3) [6, 9]. On the other hand, in the absence of disorder, the results for the localization length (transmission coefficient) and for the reflection coefficient coincide with the exact results obtained by Datta [12]. In particular, in the absence of disorder, the reflection coefficient is asymptotically constant for $N \to \infty$ [12, 13].

Now, concerning the statistical effects induced by absorption/amplification in the inverse localization lengths, our results (66a), (66b) and (68a), (68b) lead to the following conclusions:

(1) the effects are identical for absorption and for amplification, for weak as well as for strong absorption/amplification;
(2) the statistical effect induced by absorption/amplification increases the localization length for weak absorption/amplification;
(3) localization by disorder is destroyed in the presence of sufficiently strong absorption/amplification.

On the other hand, our results in (67*a*), (67*b*) and (69*a*), (69*b*) for the statistical effects induced by absorption/amplification in the reflection coefficient reveal that:

(1) weak absorption induces weak asymptotic statistical growth of $\langle |r|^2\rangle$, while corresponding weak amplification leads to no statistical effect;
(2) for large absorption/amplification parameters $|b|$, absorption and amplification induce identical weak statistical growth terms (to leading order in $|b|^{-2}$) in $\langle |r|^2\rangle$.

4. Concluding remarks

The main results of this paper are summarized in the analytical expressions (51) and (52), and (66*a*), (66*b*)–(69*a*), (69*b*) for inverse localization lengths and logarithmic reflection coefficients in short and long random tight-binding systems with absorption or gain. These results are discussed in detail in the main text. Our analysis in section 2 is valid for $N\varepsilon_0^2 \ll 1$, which characterizes the weak localization regime identified more generally by the limit $L \ll \xi_0$. It would be interesting in future work to study the effect of absorption or amplification in the strong localization (or localized) regime $L \gg \xi_0$ for weak disorder. Of special interest would be the study of the additional effects associated with anomalies in ξ_0 existing at special energies, in particular at the band centre [21]. The study of transmission and reflection in the localized regime requires a more involved treatment of the disorder, respecting, in particular, the asymptotic unitarity limit of the reflection coefficient in the absence of absorption or gain. A simple analytic treatment of statistical properties of the transmittance in the localized regime in the absence of absorption has been discussed recently in [22].

We close with brief remarks on the respective roles of different symmetries of the transfer matrix (or of the lack of them) for the disordered tight-binding system with absorption or gain studied above. In section 2, a central role is played by the transfer matrix $\hat{Q}_n$ of an elementary disordered segment enclosing just one site n. $\hat{Q}_n$ obeys the property

$$\det \hat{Q}_n = 1, \tag{70}$$

which leads to the relations

$$|t_n^{++}|^2 = |t_n^{--}|^2,\ |r_n^{+-}|^2 = |r_n^{-+}|^2, \tag{71}$$

for the reflection and transmission coefficients for waves incident from the left and from the right, respectively. As shown in section 2, (70) implies that $\det \hat{Q} = 1$, which in turn leads to the identity of the transmission coefficients $|t^{++}|^2$ and $|t^{--}|^2$ (equation (17)) for a system of N sites. Now, (71) may be viewed simply as reflecting the symmetry of the piecewise defined solutions of the Schrödinger equation for plane waves incident from the right and from the left, respectively, in a single-site random segment. This finally shows that the transfer matrix $\hat{Q}$ embodies the basic left–right symmetry of equation (2), via equation (16), which leads to the properties $\langle |t^{++}|^2\rangle = \langle |t^{--}|^2\rangle \equiv \langle |t|^2\rangle$, $\langle |r^{+-}|^2\rangle = \langle |r^{-+}|^2\rangle \equiv \langle |r|^2\rangle$ for the observable transmission and reflection coefficients.

In the absence of absorption or gain, the disordered system (2) possesses a further well-known symmetry, namely time-reversal symmetry. This symmetry implies that the 2×2 transfer matrix $\hat{X}$ satisfies the condition

$$\hat{X}^* = \sigma\hat{X}\sigma, \qquad \sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{72}$$

This symmetry is broken when absorption or gain is present as follows, for example, from the transfer matrix (23) for the pure system. Indeed, for $E = 0$, (72) would require that

$$B_N^* = C_N \qquad \text{and} \qquad A_N^* = D_N, \tag{73}$$

which is not the case for the elements (40)–(42). Now it is readily seen that the lack of time-reversal symmetry, as shown by the violation of (73), is related to a physical fact, namely the absence of current conservation, which means that $|r|^2 + |t|^2 \neq 1$. Indeed, from (17) and (29) we have, in the present case,

$$|r|^2 + |t|^2 = \frac{1 + |B_N|^2}{|D_N|^2} \neq 1, \tag{74}$$

as seen from (28), since $A_N \neq D_N^*$ and $C_N \neq B_N^*$. Note also a further related consequence of the lack of time-reversal symmetry of the perfectly absorbing or amplifying systems: this is the violation of the duality relation for the scattering matrix derived by Paaschens *et al* [9]. Violation of the duality relation of [9] for the S-matrix is easily demonstrated by substituting the transfer matrix elements (41)–(43) in equations (14) and (15) for the reflection and transmission amplitude coefficients.

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